#7 on review

7. Find the derivative matrix of the function below.

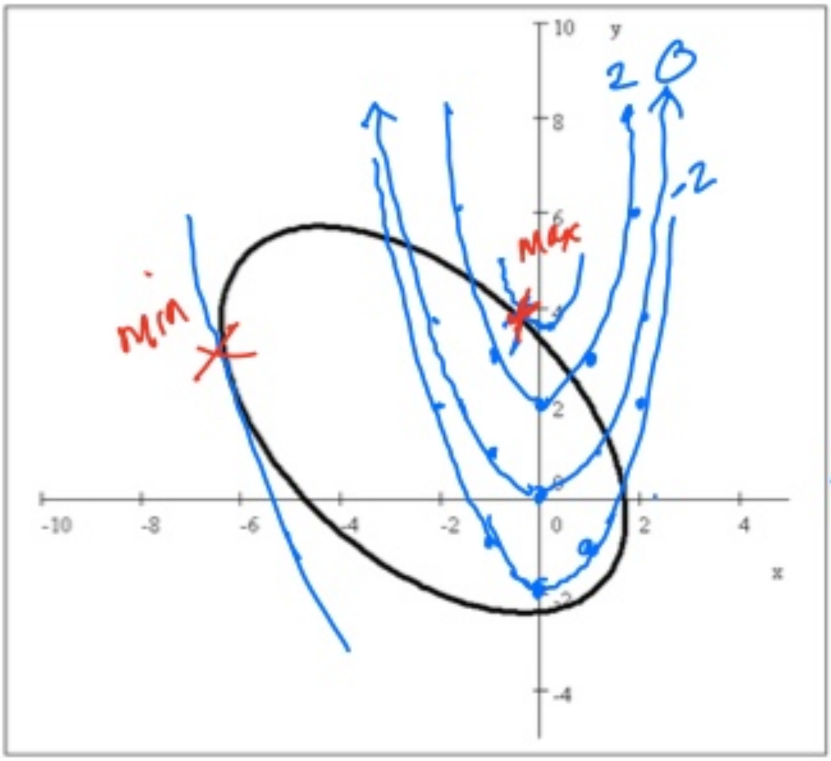
$$p(x,y) = \begin{pmatrix} 2x - \sqrt{3yx} \\ 2x3^{xy} \\ \arcsin\left(\frac{x}{y}\right) \end{pmatrix}$$

$$p(x,y) = \begin{pmatrix} p_1(x,y) \\ p_2(x,y) \\ p_3(x,y) \end{pmatrix}$$

$$p' = \begin{pmatrix} \nabla p_1 \\ \nabla p_2 \\ \nabla p_3 \end{pmatrix}$$

$$p' = \begin{pmatrix} 2 - \sqrt{3y}\left(\frac{1}{2}x^{-1/2}\right) & -\sqrt{3x}\,\frac{1}{2}y^{-1/2} \\ \cdots & \cdots \\ \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \left(\frac{1}{y}\right) & \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\left(-\frac{1}{y^2}\right) \end{pmatrix}$$

11. Find/estimate the absolute maximum and minimum of $F(x,y)=y-x^2$ restricted to the curve pictured below.



Contours of $y-x^2$

$y-x^2=C$

$\Leftrightarrow y=x^2+C$

$y-x^2=0$

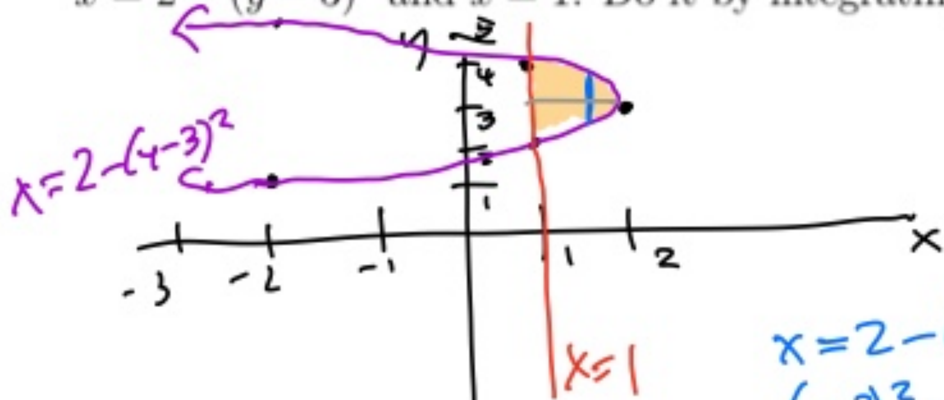
$y=x^2$

min $=(-6.2,\ 3.2)$

max $=(-.5,\ 3.9)$

(14)

(b) $\displaystyle\int_R \frac{y}{x}\, dA$, where R is the region between the parabola $x=2-(y-3)^2$ and $x=1$. Do it by integrating y first.



$x=2-(y-3)^2$

$(y-3)^2=2-x$

$y-3=\pm\sqrt{2-x}$

$y=3\pm\sqrt{2-x}$

$3-\sqrt{2-x} \le y \le 3+\sqrt{2-x}$

$1\le x\le 2$

$$\boxed{\int_1^2\int_{3-\sqrt{2-x}}^{3+\sqrt{2-x}} \frac{y}{x}\,dy\,dx}$$

Did x-first:

$1\le x\le 2-(y-3)^2$

$2\le y\le 4$

$$\int_2^4\int_1^{2-(y-3)^2} \frac{y}{x}\,dx\,dy$$

$$\hookrightarrow = \int_1^2 \left(\frac{y^2}{2x} \Big|_{y=3-\sqrt{2-x}}^{3+\sqrt{2-x}} \right) dx$$

$$= \int_1^2 \frac{1}{2x} \Big(\underbrace{\underbrace{(3+\sqrt{2-x})^2}_{\substack{9+(2-x)\\ +6\sqrt{2-x}}} - \underbrace{(3-\sqrt{2-x})^2}_{\substack{9+2-x\\ -6\sqrt{2-x}}}}_{12\sqrt{2-x}} \Big) dx$$

$$= \int_1^2 \frac{12\sqrt{2-x}}{2x} dx = \int_1^2 \frac{6\sqrt{2-x}}{x} dx$$

Possible substitutions

1. $x = 2\sin^2\theta$

   $dx = 4\sin\theta\cos\theta\, d\theta$

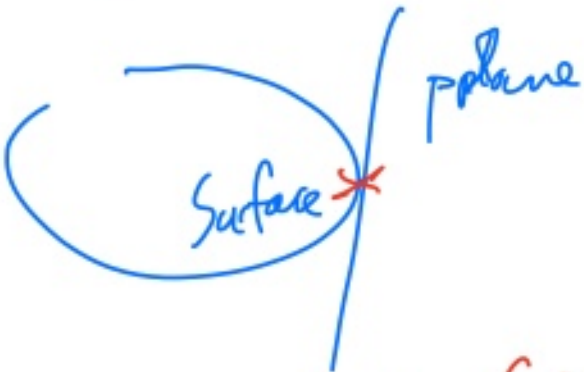
2. $2-x = u^2 \Rightarrow x = 2-u^2$

   $-dx = 2u\,du$

   $dx = -2u\,du$

6. You wish to find all points of the surface $\frac{z+2y}{y^2-x^2} = -x-1$ where the tangent plane is parallel to the plane $x - y + z = 2024$. Set up the equations that need to be solved, and show the sagemath code that would be used to solve it. [You should be able to reduce it to solving two equations and two unknowns.]

Want normal vectors to be parallel



$$z + 2y = -(x+1)(y^2 - x^2)$$

$$F(x,y,z) = z + 2y + (x+1)(y^2 - x^2) = 0$$

$$\nabla F = \left((y^2 - x^2) + (x+1)(-2x),\ 2 + (x+1)(2y),\ 1\right)$$

$-2x^2 - 2x$

$y^2 - 3x^2 - 2x$

$2 + 2xy + 2y$

plane $x - y + z = 2024$

$\nabla \text{plane} = (1, -1, 1)$

$$y^2 - 3x^2 - 2x = 1 \cdot c$$

$$2 + 2xy + 2y = (-1) \cdot c$$

$$1 = 1 \cdot c$$

$$y^2 - 3x^2 - 2x = 1,\quad 2 + 2xy + 2y = -1.$$

$1 + xy + y = -\frac{1}{2}$

$(x+1)y = -\frac{3}{2}$

$y = \frac{-3}{2(x+1)}$

↳ Sagemath

```
f(x,y) = y^2-3*x^2-2*x
g(x,y) = 2+2*x*y+2*y
eq1 = f(x,y)==1
eq2 = g(x,y)==-1
solve([eq1,eq2],x,y)
```

$$\int_0^1 \int_x^2 3xy^2 \, dy \, dx$$

```
f(x,y) = 3*x*y^2
show(integral(integral(f(x,y),y,x,2),
         x,0,1))
```

numerical .n()

Derivatives

$$\frac{\partial^2}{\partial x \partial y}\left(\sin(x)\cdot\arccos(e^y)\right)$$

```
f(x,y) = sin(x)*arccos(e^y)
show( diff(f(x,y),x,y))
```

12. A cylindrical can is designed to hold 14 fluid ounces of beans. The top and bottom disks of the container cost 1.5 cents per square inch, the cylinder part of the container costs 1.4 cents per square inch, and the circumference of the top and of the bottom costs 3.1 cents per inch (where the metal is doubled and creased). You want to choose the dimensions of this container so that the cost is minimized. Set up the equations that would be used to solve this problem using Lagrange multipliers. [Additional info: 1 fl oz = 1.805 in^3.]

$14 \text{ fl oz} = 14(1.805) \text{ in}^3$

$V(r,h) = \pi r^2 h = 14(1.805)$

constraint

function:

$F(r,h) = \text{cost}.$

$F(r,h) = (2\pi r^2)(1.5) + (2\pi r h)(1.4) + (4\pi r)(3.1)$

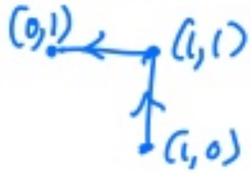
$$\begin{cases} \nabla F = \lambda \nabla V \\ g = 14(1.805) \end{cases}$$

Warm-up ① Find $\int_C 2x\cos(y)\,dx - x^2\sin(y)\,dy$

if C is a curve from $(1,0)$ to $(0,1)$

where C is ⓐ along unit circle

ⓑ along a straight line

ⓒ Along this path 

② Same for $\int_C x^2\,dy$

Line Integrals of Vector Fields over curves $\Longleftrightarrow$ integrals of 1-forms over curves

$$\oint_\alpha V\cdot ds = \int_\alpha V_1\,dx + V_2\,dy$$

$$\parallel$$

$$\int V(\alpha(t))\cdot \alpha'(t)\,dt$$

$V(x,y) = (V_1(x,y), V_2(x,y))$

$\alpha(t) = (x(t), y(t))$

$\alpha'(t) = (x'(t), y'(t))$

$$V_1(x,y)\,dx + V_2(x,y)\,dy$$

plug in $\alpha(t)$

$$V_1(x(t),y(t))\,x'(t)\,dt + V_2(x(t),y(t))\,y'(t)\,dt$$

$$= V(\alpha(t))\cdot \alpha'(t)\,dt$$

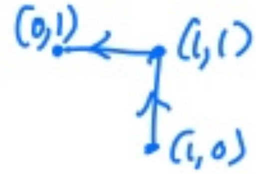
① Find $\int_C 2x\cos(y)\,dx - x^2\sin(y)\,dy$

if C is a curve from (1,0) to (0,1)

where C is ⓐ along unit circle

ⓑ along a straight line

ⓒ Along this path



② Same for $\int_C x^2\,dy$

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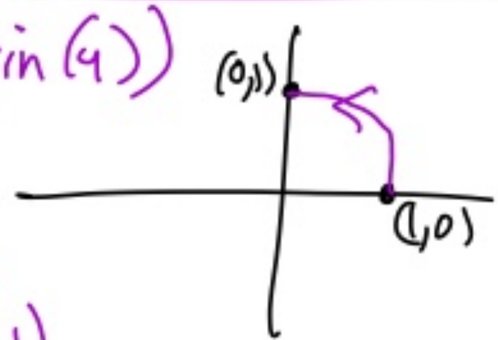
① $V(x,y) = (2x\cos(y), -x^2\sin(y))$



ⓐ unit circle path from (1,0) to (0,1)

$$\alpha(t) = (\overset{x(t)}{\cos(t)}, \overset{y(t)}{\sin(t)})$$

$$0 \le t \le \frac{\pi}{2}$$

$$\alpha'(t) = (-\sin(t), \cos(t))$$

$$\Rightarrow \oint_\alpha V\cdot ds = \int_0^{\pi/2} V(\alpha(t))\cdot\alpha'(t)\,dt$$

$$= \int_0^{\pi/2} \left(2(\cos(t))\cos(\sin(t)), -\cos^2(t)\cdot\sin(\sin(t))\right) \cdot (-\sin(t), \cos(t))\,dt$$

$$= \int_0^{\pi/2} \left[-2\,\underline{\cos(t)\sin(t)}\cos(\sin(t)) + -\underline{\cos^3(t)}\sin(\sin(t))\right] dt$$

$$= \quad u=\sin(t) \qquad t=0 \;\; u=0$$
$$du=\underline{\cos(t)\,dt} \qquad t=\tfrac{\pi}{2} \;\; u=1$$

$$= \int_0^1 -2u\cos(u)\,du + -\underbrace{\cos^2(t)}_{\boxed{\begin{matrix}1-\sin^2\\ 1-u^2\end{matrix}}}\sin(u)\,du$$

$$(u^2-1)$$

$$= \int_0^1 -2u\cos(u)\,du + (u^2-1)\sin(u)\,du$$

$$= -2\int_0^1 \underbrace{u\cos(u)\,du}_{\text{parts}} + \int \underbrace{(u^2-1)\sin(u)\,du}$$

parts

$U=u \quad dU=du$

$dV=\cos(u)\,du \quad V=\sin(u)$

$U=(u^2-1) \quad dU=2u\,du$

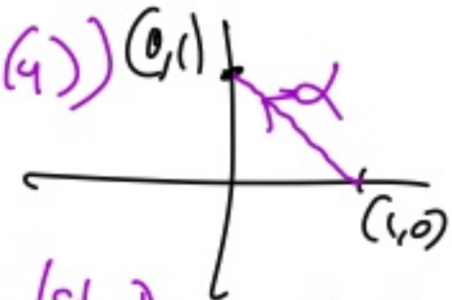
$dV=\sin(u)\,du \quad V=-\cos(u)$

...

$= \ldots$ $\boxed{\text{Ans} = -1}$

(b) along straight line:

$$V(x,y) = \left(2x\cos(y),\, -x^2\sin(y)\right)$$



$$\oint_\alpha V\cdot ds$$

$\alpha(t) = (\text{start})(1-t) + (\text{end})t$

$\alpha(t) = (1,0)(1-t) + (0,1)t$

$= (1-t,\, t)$

$\alpha'(t) = (-1, 1)_{0 \le t \le 1}$

$$\int_{t=0}^{1} V(\alpha(t)) \cdot \alpha'(t)\, dt$$

$$= \int_0^1 V(\overset{x}{1-t}, \overset{y}{t}) \cdot (-1, 1)\, dt$$

$$= \int_0^1 \left(2(1-t)\cos(t), -(1-t)^2 \sin(t)\right) \cdot (-1, 1)\, dt$$

$$= \int_0^1 \left(-2(1-t)\cos(t) - (1-t)^2 \sin(t)\right) dt$$

integrate by parts ...

$\ldots \boxed{\text{ans} = -1}$